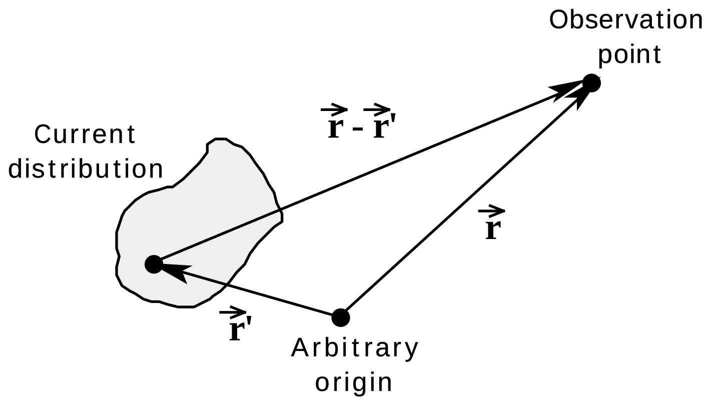
An Introduction to Antenna Theory: Radiation from a Hertzian Dipole

The general "radiation" or "antenna" problem comes down to the task of finding the electromagnetic field associated with or derived from a given or known current distribution.



The answer is straight forward in concept. The vector potential of the total field is just a weighted superposition of spherical wave contributions (see lecture entitled *Electromagnetic Radiation - The Basics* for a treatment of spherical waves) from each current element of the distribution -- *viz*.

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = \frac{\mu}{4\pi}\sum_{i=1}^{N}\left\{\left[\iiint_{\Delta R_i \to 0}\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\,dV\right]\frac{\exp\left[-j\,k\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}_i\right|\right]}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}_i\right|}\right\}$$
$$= \frac{\mu}{4\pi}\iiint\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\,\frac{\exp\left[-j\,k\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|\right]}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|}\,dV' \qquad \text{[I-1]}$$

<u>There are two difficulties</u>:

1. The computations are messy. Fortunately, even with modest computational resources, this is no longer an insuperable problem.
2. The real problem is that, in general, **we do not really know the current distribution!**

As always, intuition is important and it is very helpful to study in detail one simple distribution -- *i.e.* a Hertzian dipole (an elemental or infinitesimal current element) located at the origin and oscillating in the z-direction.

$$
\begin{aligned}
\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) &= \frac{\mu}{4\pi}\iiint \vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\,\frac{\exp\left[-j\,k\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|\right]}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|}\,dV' \\
&\approx \frac{\mu}{4\pi}\,\frac{\exp\left[-j\,k\left|\vec{\mathbf{r}}\right|\right]}{\left|\vec{\mathbf{r}}\right|}\iiint_{\mathrm{vol}\to 0}\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\,dV' \\
&\approx \frac{\mu}{4\pi}\,\frac{\exp\left[-j\,k\,\mathrm{r}\right]}{\mathrm{r}}\left[\hat{\mathbf{z}}\,I(\omega)\,\delta\,\ell\right]
\end{aligned}
\qquad [\text{I-2}]
$$

Given this relatively simple expression for the vector potential, the arduous, but straightforward, task is finding the associated **electric** and **magnetic** field strengths. From the above expression for the **vector potential**

$$
\begin{aligned}
\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) &= \frac{1}{\mu}\,\vec{\nabla}\times\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) \\
&\approx \frac{1}{\mu}\,\vec{\nabla}\times\left\{\frac{\mu}{4\pi}\,\frac{\exp\left[-j\,k\,\mathrm{r}\right]}{\mathrm{r}}\left[\hat{\mathbf{z}}\,I(\omega)\,\delta\,\ell\right]\right\} \\
&\approx \frac{I(\omega)\,\delta\,\ell}{4\pi}\,\vec{\nabla}\left\{\frac{\exp\left[-j\,k\,\mathrm{r}\right]}{\mathrm{r}}\right\}\times\hat{\mathbf{z}} \\
&\approx \frac{I(\omega)\,\delta\,\ell}{4\pi}\,\frac{d}{d\,\mathrm{r}}\left\{\frac{\exp\left[-j\,k\mathrm{r}\right]}{\mathrm{r}}\right\}\vec{\nabla}\,\mathrm{r}\times\hat{\mathbf{z}}
\end{aligned}
\qquad [\text{I-3}]
$$

and since $\vec{\nabla}\,\mathrm{r} = \dfrac{\vec{\mathbf{r}}}{\mathrm{r}} = \hat{\mathbf{r}}$ (see lecture entitled *Electromagnetic Radiation - The Basics*)

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \approx \frac{I(\omega)\,\delta\ell}{4\pi}\frac{d}{d\mathrm{r}}\left\{\frac{\exp[-j\,k\mathrm{r}]}{\mathrm{r}}\right\}\hat{\mathbf{r}}\times\hat{\mathbf{z}}$$

$$\approx -\frac{I(\omega)\,\delta\ell}{4\pi}\left\{\frac{j\,k}{\mathrm{r}}+\frac{1}{\mathrm{r}^2}\right\}\exp[-j\,k\,\mathrm{r}]\,\hat{\mathbf{r}}\times\hat{\mathbf{z}} \qquad \text{[I-4]}$$

The electric field strength is obtained most directly by using the "Ampère's law" Maxwell equation -- *viz*.

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \frac{1}{j\omega\varepsilon}\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = \frac{\eta}{jk}\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-5]}$$

where $\eta = \sqrt{\mu/\varepsilon}$ -- and thus

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \approx j\frac{\eta\, I(\omega)\,\delta\ell}{4\pi k}\vec{\nabla}\times\left\{\left[\frac{j\,k}{\mathrm{r}}+\frac{1}{\mathrm{r}^2}\right]\exp[-j\,k\,\mathrm{r}]\,\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right\}$$

$$\approx j\frac{\eta\, I(\omega)\,\delta\ell}{4\pi k}\left\{\left[\frac{j\,k}{\mathrm{r}}+\frac{1}{\mathrm{r}^2}\right]\exp[-j\,k\,\mathrm{r}]\,\vec{\nabla}\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]+\vec{\nabla}\left[\left[\frac{j\,k}{\mathrm{r}}+\frac{1}{\mathrm{r}^2}\right]\exp[-j\,k\,\mathrm{r}]\right]\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]\right\} \qquad \text{[I-6]}$$

$$\approx j\frac{\eta\, I(\omega)\,\delta\ell}{4\pi k}\left\{\left[\frac{j\,k}{\mathrm{r}}+\frac{1}{\mathrm{r}^2}\right]\exp[-j\,k\,\mathrm{r}]\,\vec{\nabla}\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]+\frac{d}{d\mathrm{r}}\left[\left[\frac{j\,k}{\mathrm{r}}+\frac{1}{\mathrm{r}^2}\right]\exp[-j\,k\,\mathrm{r}]\right]\vec{\nabla}\mathrm{r}\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]\right\}$$

by the "bac-cab" rule[1]

$$\vec{\nabla}\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}] = \hat{\mathbf{z}}\,\vec{\nabla}\cdot\hat{\mathbf{r}} - \hat{\mathbf{z}}\cdot\vec{\nabla}\,\hat{\mathbf{r}}$$

$$= \frac{\partial}{\partial z}\left(\frac{\vec{\mathbf{r}}}{\mathrm{r}}\right) - \hat{\mathbf{z}}\,\vec{\nabla}\cdot\left(\frac{\vec{\mathbf{r}}}{\mathrm{r}}\right) = \frac{\partial}{\partial z}\left(\frac{\vec{\mathbf{r}}}{\mathrm{r}}\right) - \hat{\mathbf{z}}\,\vec{\mathbf{r}}\cdot\vec{\nabla}\left(\frac{1}{\mathrm{r}}\right) - \frac{\hat{\mathbf{z}}}{\mathrm{r}}\vec{\nabla}\cdot\vec{\mathbf{r}} \qquad \text{[I-7a]}$$

$$= \left(\frac{1}{\mathrm{r}}\right)\hat{\mathbf{z}} - \left(\frac{z}{\mathrm{r}^3}\right)\vec{\mathbf{r}} + \left(\frac{\vec{\mathbf{r}}\cdot\hat{\mathbf{r}}}{\mathrm{r}^2}\right)\hat{\mathbf{z}} - \frac{3\hat{\mathbf{z}}}{\mathrm{r}} = -\left(\frac{z}{\mathrm{r}^2}\right)\hat{\mathbf{r}} - \frac{\hat{\mathbf{z}}}{\mathrm{r}}$$

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1 That is $\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)$.

and
$$\vec{r}\times\left[\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right]=\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right]=\left(\frac{z}{r}\right)\hat{\mathbf{r}}-\hat{\mathbf{z}} \qquad \text{[I-7b]}$$

Therefore

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\approx j\frac{\eta\, I(\omega)\,\delta\ell}{4\pi k}\exp\left[-j\,k r\right]\left\{-\left[\frac{j\,k}{r}+\frac{1}{r^{2}}\right]\left[\left(\frac{z}{r^{2}}\right)\hat{\mathbf{r}}+\frac{\hat{\mathbf{z}}}{r}\right]+\left[-\frac{j\,2k}{r^{2}}-\frac{2}{r^{3}}+\frac{k^{2}}{r}\right]\left[\left(\frac{z}{r}\right)\hat{\mathbf{r}}-\hat{\mathbf{z}}\right]\right\} \qquad \text{[I-8]}$$

$$\approx j\frac{\eta\, I(\omega)\,\delta\ell}{4\pi\, k}\exp\left[-j\,k r\right]\left\{\left[\frac{j\,k}{r^{2}}+\frac{1}{r^{3}}\right]\left[\hat{\mathbf{z}}-\frac{3z}{r}\,\hat{\mathbf{r}}\right]+\frac{k^{2}}{r}\left[\left(\frac{z}{r}\right)\hat{\mathbf{r}}-\hat{\mathbf{z}}\right]\right\}$$

Given the complexity of these field expressions, it extremely useful to "parse" the terms according to their **"one-over-r-to-the-n-ness"** -- *viz*.

<u>The Radiation or Far-Zone Fields</u> ("one-over-r"):

$$\vec{\mathbf{H}}_{\mathrm{fz}}(\vec{\mathbf{r}},\omega)\approx -j\,\frac{k\, I(\omega)\,\delta\ell}{4\pi}\left[\frac{\exp\left[-j\,k r\right]}{r}\right]\hat{\mathbf{r}}\times\hat{\mathbf{z}} \qquad \text{[I-9a]}$$

$$\vec{\mathbf{E}}_{\mathrm{fz}}(\vec{\mathbf{r}},\omega)\approx j\,\frac{\eta\, k\, I(\omega)\,\delta\ell}{4\pi}\left[\frac{\exp\left[-j\,k r\right]}{r}\right]\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right] \qquad \text{[I-9b]}$$

<u>The Induction or Intermediate-Zone Fields</u> ("one-over-r-squared"):

$$\vec{\mathbf{H}}_{\mathrm{iz}}(\vec{\mathbf{r}},\omega)\approx -\frac{I(\omega)\,\delta\ell}{4\pi}\left[\frac{\exp\left[-j\,k r\right]}{r^{2}}\right]\hat{\mathbf{r}}\times\hat{\mathbf{z}} \qquad \text{[I-10a]}$$

$$\vec{\mathbf{E}}_{\mathrm{iz}}(\vec{\mathbf{r}},\omega)\approx -\frac{\eta\, I(\omega)\,\delta\ell}{4\pi}\left[\frac{\exp\left[-j\,k r\right]}{r^{2}}\right]\left[\hat{\mathbf{z}}-\frac{3(\hat{\mathbf{z}}\ \hat{\mathbf{r}})}{r}\,\hat{\mathbf{r}}\right] \qquad \text{[I-10b]}$$

<u>The Static or Near-Zone Field</u> ("one-over-r-cubed"):

$$\vec{\mathbf{E}}_{nz}(\vec{\mathbf{r}},\omega) \approx j\,\frac{\eta\, I(\omega)\,\delta\ell}{4\pi k}\left[\frac{\exp[-j\,k\,r]}{r^3}\right]\left[\hat{\mathbf{z}} - \frac{3(\hat{\mathbf{z}}\;\hat{\mathbf{r}})}{r}\,\hat{\mathbf{r}}\right] \qquad \text{[I-11]}$$

Consider now the **Poynting Vector** in the **<u>Radiation or Far-Zone</u>** since it is a measure of the radiated power far from an antenna and, thus, is the key factor in the evaluation of the characteristics and performance of antenna designs:

$$\begin{aligned}
\vec{\mathbf{S}}_{fz}(\vec{\mathbf{r}},\omega) &= \frac{1}{2}\left\{\vec{\mathbf{E}}_{fz}(\vec{\mathbf{r}},\omega)\times\vec{\mathbf{H}}_{fz}(\vec{\mathbf{r}},\omega)\right\}\\
&= \frac{1}{2}\left\{ j\,\frac{\eta\, k\, I(\omega)\,\delta\ell}{4\pi}\left[\frac{\exp[-j\,k\,r]}{r}\right]\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]\right\}\\
&\qquad\times\left\{ j\,\frac{k\,I\;(\omega)\,\delta\ell}{4\pi}\left[\frac{\exp[j\,k\,r]}{r}\right]\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right\}\\
&= \frac{1}{2}\,\frac{\eta\, k^2\,|I(\omega)|^2\,[\delta\ell]^2}{[4\pi]^2\; r^2}\left\{[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]\times[\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\hat{\mathbf{z}}]]\right\}\\
&= \frac{1}{2}\left[\frac{1}{4\pi}\right]^2\frac{\eta\, k^2\,|I(\omega)|^2\,[\delta\ell]^2}{r^2}\,|\hat{\mathbf{r}}\times\hat{\mathbf{z}}|^2\;\hat{\mathbf{r}}\\
&= \frac{1}{2}\left[\frac{1}{4\pi}\right]^2\frac{\eta\, k^2\,|I(\omega)|^2\,[\delta\ell]^2}{r^2}\,\sin^2\theta\;\hat{\mathbf{r}}
\end{aligned} \qquad \text{[I-12]}$$

This expression may be integrated over a spherical surface of radius r to obtain the **total power radiated** by the infinitesimal current element

$$
\begin{aligned}
W_{fz}(\vec{\mathbf{r}},\omega) &= \int_0^{\pi} \vec{\mathbf{S}}_{fz}(\vec{\mathbf{r}},\omega)\cdot\hat{\mathbf{r}}\; 2\pi\, r^2 \sin\theta\, d\theta \\
&= \frac{1}{2}\,\frac{\eta\, k^2\, |I(\omega)|^2\,[\delta\ell]^2}{8\pi}\int_0^{\pi}\sin^3\theta\, d\theta
\end{aligned}
\qquad [I-13]
$$

and since $\displaystyle\int_0^{\pi}\sin^3\theta\, d\theta = \int_0^{\pi}\sin\theta\left[1-\cos^2\theta\right] d\theta = -\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{4}{3}$

$$
\begin{aligned}
W_{fz}(\vec{\mathbf{r}},\omega) &= \frac{1}{2}\,\frac{\eta\, k^2\, |I(\omega)|^2\,[\delta\ell]^2}{6\pi} \\
&= \frac{1}{2}\,\eta\,\frac{2\pi\,|I(\omega)|^2}{3}\left[\frac{\delta\ell}{\lambda}\right]^2 \\
&= \frac{1}{2}\, R_{rad}\,|I(\omega)|^2
\end{aligned}
\qquad [I-14]
$$

where $\displaystyle R_{rad} \equiv \frac{\eta\, k^2\,[\delta\ell]^2}{6\pi} = \eta\,\frac{2\pi}{3}\left[\frac{\delta\ell}{\lambda}\right]^2$ is defined as the **radiation resistance**. Further, since the impedance of free space $\eta \equiv \sqrt{\mu_0/\varepsilon_0} \approx 120\pi$ ohms

$$
\begin{aligned}
W_{fz}(\vec{\mathbf{r}},\omega) &\approx \frac{1}{2}(120\pi)\,\frac{2\pi\,|I(\omega)|^2}{3}\left[\frac{\delta\ell}{\lambda}\right]^2 \text{ watts/amp}^2 \\
&\approx 40\pi^2\,|I(\omega)|^2\left[\frac{\delta\ell}{\lambda}\right]^2 \text{ watts/amp}^2
\end{aligned}
\qquad [I-15a]
$$

and

$$
\begin{aligned}
R_{rad} &\approx (120\pi)\,\frac{2\pi}{3}\left[\frac{\delta\ell}{\lambda}\right]^2 \text{ ohms} \\
&\approx 80\pi^2\left[\frac{\delta\ell}{\lambda}\right]^2 \text{ ohms} \approx 790\left[\frac{\delta\ell}{\lambda}\right]^2 \text{ ohms}
\end{aligned}
\qquad [I-15b]
$$

A Simple Application - Light Scattering: As a model of an electromagnetic scattering center we suppose that we have a very small, polarizable "molecule" -- *viz*.

$$\vec{\mathbf{p}}(\omega) = \alpha(\omega)\,\vec{\mathbf{E}}_{\rm inc}(\omega) \qquad \text{[I-16]}$$

where $\alpha(\omega)$ is the "polarizability" of the molecule and, thus, the current flow in the molecule is given by

$$I(\omega)\,\delta\ell\,\hat{\mathbf{z}} = j\,\omega\,\vec{\mathbf{p}}(\omega) = j\,\omega\,\alpha(\omega)\,\vec{\mathbf{E}}_{\rm inc}(\omega) \qquad \text{[I-17]}$$

When this induced current is substituted into Equation [I-12], we obtain the electromagnetic intensity re-radiated or scattered by the molecule

$$\begin{aligned}\vec{\mathbf{S}}^{\rm scat}_{\rm fz}(\vec{\mathbf{r}},\omega) &= \frac{1}{2}\left[\frac{1}{4\pi}\right]^2 \frac{\eta\,k^2\,|I(\omega)|^2\,[\delta\ell]^2}{\mathrm{r}^2}\,\sin^2\theta\,\hat{\mathbf{r}}\\ &= \frac{1}{2}\left[\frac{1}{4\pi}\right]^2 \frac{\eta\,k^2}{\mathrm{r}^2}\left[\omega^2\,\alpha^2(\omega)\left|\vec{\mathbf{E}}_{\rm inc}(\omega)\right|^2\right]\sin^2\theta\,\hat{\mathbf{r}} \qquad \text{[I-18]}\\ &= \frac{1}{2}\left[\frac{c}{4\pi}\right]^2 \frac{\eta\,k^4}{\mathrm{r}^2}\left[\alpha^2(\omega)\left|\vec{\mathbf{E}}_{\rm inc}(\omega)\right|^2\right]\sin^2\theta\,\hat{\mathbf{r}}\end{aligned}$$

This is the very famous **Rayleigh scattering** (*why the sky is blue*) **formula** which states that the scattering intensity varies as **"one-over-lamda-to-the-fourth-power."**

APPENDIX: ON SPHERICAL WAVES

We need to establish that spherical waves are valid solutions of Maxwell's equation (or, more precisely, of the inhomogeneous Helmholtz equation derived from Maxwell's equations)

$$\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) + k^2\, \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = -\mu\, \vec{\mathbf{J}}(\vec{\mathbf{r}},\omega) \qquad \text{[A-1]}$$

1. Let us first look for solutions in "current-free" region so that

$$\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) + k^2\, \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[A-2]}$$

Our goal is to find a solution which depends only on the magnitude of the observer's position vector and, thus, we look for solutions in the form

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{a}}\, \frac{\mathrm{f}(\mathrm{r},\omega)}{\mathrm{r}} \qquad \text{[A-3]}$$

where $\mathrm{r} = |\vec{\mathbf{r}}|$ and $\vec{\mathbf{a}}$ is a constant vector. To see if this form of solution works, we need to find div grad $\left[\frac{\mathrm{f}(\mathrm{r},\omega)}{\mathrm{r}}\right]$. To that end, we first use the "chain rule" to find grad $\left[\frac{\mathrm{f}(\mathrm{r},\omega)}{\mathrm{r}}\right]$ -- *viz*.

$$\begin{aligned} \mathrm{grad}\left[\frac{\mathrm{f}(\mathrm{r},\omega)}{\mathrm{r}}\right] &= \vec{\nabla}\left[\frac{\mathrm{f}(\mathrm{r},\omega)}{\mathrm{r}}\right] = \left[\vec{\nabla}\,\mathrm{r}\right]\frac{d}{d\mathrm{r}}\left[\frac{\mathrm{f}(\mathrm{r},\omega)}{\mathrm{r}}\right] \\ &= \left[\vec{\nabla}\,\mathrm{r}\right]\left[\frac{1}{\mathrm{r}}\frac{d}{d\mathrm{r}}\mathrm{f}(\mathrm{r},\omega) - \frac{1}{\mathrm{r}^2}\,\mathrm{f}(\mathrm{r},\omega)\right] \end{aligned} \qquad \text{[A-4]}$$

However

$$\begin{aligned}\nabla r &= \nabla\left[\sqrt{x^2+y^2+z^2}\right]\\ &= \frac{x\,\hat{\mathbf{x}}}{\sqrt{x^2+y^2+z^2}}+\frac{y\,\vec{\mathbf{y}}}{\sqrt{x^2+y^2+z^2}}+\frac{z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} \qquad \text{[A-5]}\\ &= \frac{\vec{\mathbf{r}}}{r}=\hat{\mathbf{r}}\end{aligned}$$

$$\begin{aligned}\nabla^2\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) &= \nabla\cdot\nabla\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)=\nabla\cdot\nabla\left[\vec{\mathbf{a}}\,\frac{f(r,\omega)}{r}\right]\\ &= \vec{\mathbf{a}}\,\nabla\cdot\nabla\left[\frac{f(r,\omega)}{r}\right] \qquad \text{[A-6]}\\ &= \vec{\mathbf{a}}\,\nabla\cdot\left\{\hat{\mathbf{r}}\left[\frac{1}{r}\frac{d}{dr}f(r,\omega)-\frac{1}{r^2}f(r,\omega)\right]\right\}\end{aligned}$$

and using the "chain rule"

$$\begin{aligned}\nabla^2\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) &= \vec{\mathbf{a}}\,\nabla\cdot\left\{\hat{\mathbf{r}}\left[\frac{1}{r}\frac{d}{dr}f(r,\omega)-\frac{1}{r^2}f(r,\omega)\right]\right\}\\ &= \vec{\mathbf{a}}\left\{\left[\nabla r\; r\,\hat{\mathbf{r}}\right]\frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}f(r,\omega)-\frac{1}{r^3}f(r,\omega)\right]\right.\\ &\qquad\left.+3\left[\frac{1}{r^2}\frac{d}{dr}f(r,\omega)-\frac{1}{r^3}f(r,\omega)\right]\right\}\\ &= \vec{\mathbf{a}}\left\{r\left[\frac{1}{r^2}\frac{d^2}{dr^2}f(r,\omega)-\frac{3}{r^3}\frac{d}{dr}f(r,\omega)+\frac{3}{r^4}f(r,\omega)\right]\right. \qquad \text{[A-7]}\\ &\qquad\left.+3\left[\frac{1}{r^2}\frac{d}{dr}f(r,\omega)-\frac{1}{r^3}f(r,\omega)\right]\right\}=\vec{\mathbf{a}}\left[\frac{1}{r}\frac{d^2}{dr^2}f(r,\omega)\right]\end{aligned}$$

Therefore $\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) + k^2\, \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = 0$ becomes

$$\vec{\mathbf{a}} \left[\frac{1}{r} \frac{d^2}{d\,r^2} f(r,\omega) \right] + k^2 \left[\vec{\mathbf{a}} \frac{f(r,\omega)}{r} \right] = 0 \qquad \text{[A-8a]}$$

or

$$\frac{d^2}{d\,r^2} f(r,\omega) + k^2\, f(r,\omega) = 0 \qquad \text{[A-8b]}$$

Therefore, we see that the homogeneous Helmholtz equation has two independent solutions -- *viz.*

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{a}}\, C_+ \left[\frac{\exp[-j\,k\,r]}{r} \right] + \vec{\mathbf{a}}\, C_- \left[\frac{\exp[+j\,k\,r]}{r} \right] \qquad \text{[A-9]}$$

$\vec{\mathbf{a}}\, C_+ \left[\frac{\exp[-j\,k\,r]}{r} \right]$ Outwardly propagating spherical wave

$\vec{\mathbf{a}}\, C_- \left[\frac{\exp[+j\,k\,r]}{r} \right]$ Inwardly propagating spherical wave

2. To determine the constant in the outwardly propagating spherical wave, **<u>we now study the behavior of the inhomogeneous equation in the vicinity of the singularity</u>** -- *i.e.* at r = 0 -- where the source of the wave must be located. To cope with the singularity, we integrate the inhomogeneous equation over a small sphere of radius R centered at r = 0.

$$\iiint_{\text{vol. of sphere}} \nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)\, dV + k^2 \iiint_{\text{vol. of sphere}} \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)\, dV = -\mu \iiint_{\text{vol. of sphere}} \vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\, dV \qquad \text{[A-10]}$$

If we use Gauss' theorem to transform the first term on the left hand side

$$\iint_{\text{surf. of sphere}} \left[d\vec{\mathbf{S}} \cdot \vec{\nabla}\right]\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) + k^2 \iiint_{\text{vol. of sphere}} \vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)\,dV = -\mu \iiint_{\text{vol. of sphere}} \vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\,dV \qquad \text{[A-11]}$$

and substitute the outwardly propagating spherical wave form

$$\iint_{\text{surf. of sphere}} \left[d\vec{\mathbf{S}} \cdot \vec{\nabla}\mathrm{r}\right]\frac{d}{d\mathrm{r}}\left\{C_+\left[\frac{\exp[-j\,k\,\mathrm{r}]}{\mathrm{r}}\right]\right\} + k^2 \iiint_{\text{vol. of sphere}} \left\{C_+\left[\frac{\exp[-j\,k\,\mathrm{r}]}{\mathrm{r}}\right]\right\}dV = -\mu \iiint_{\text{vol. of sphere}} \left[\vec{\mathbf{a}} \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\right]dV \qquad \text{[A-12a]}$$

$$C_+ \iint_{\text{surf. of sphere}} \frac{d}{d\mathrm{r}}\left[\frac{\exp[-j\,k\,\mathrm{r}]}{\mathrm{r}}\right]R^2\,d\Omega + k^2\,C_+ \iiint_{\text{vol. of sphere}} \left\{\left[\frac{\exp[-j\,k\,\mathrm{r}]}{\mathrm{r}}\right]\right\}dV = -\mu \iiint_{\text{vol. of sphere}} \left[\vec{\mathbf{a}} \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\right]dV \qquad \text{[A-12b]}$$

Therefore in the limit that the <u>radius of the small sphere goes to zero</u>

$$-4\pi\,C_+ = -\mu \iiint_{\substack{\text{vol. of sphere}\\ R\to 0}} \left[\vec{\mathbf{a}} \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\right]dV \qquad \text{[A-13]}$$

so that

$$\boxed{\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = \frac{\mu}{4\pi}\left\{\iiint_{R\to 0}\left[\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)\right]dV\right\}\frac{\exp[-j\,k\,\mathrm{r}]}{\mathrm{r}}} \qquad \text{[A-14]}$$