Guided Wave Formulation of Maxwell's Equations

I. General Theory:

**Recapitulation -- frequency domain formulation of the macroscopic Maxwel
l equations in a *source-free* region:**

$$\text{curl}\ \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -j\omega\mu\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-1a]}$$

$$\text{curl}\ \vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = +j\omega\varepsilon\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-1b]}$$

$$\text{div}\ \varepsilon\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-1c]}$$

$$\text{div}\ \mu\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-1d]}$$

Important Review:
Let us once again derive the electromagnetic wave equations (or more precisely, the electro
magnetic Helmholtz equations) by starting with Eq. [I-1a]. as we have shown many time
s, the trick is to operate on this equation with the **curl** operator -- *viz*.

$$\text{curl curl}\ \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -j\omega\,\text{curl}\left[\mu\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)\right] = -j\omega\mu\left[+j\omega\varepsilon\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[I-2]}$$

and then use the "abc = bac - cab" rule† - *i.e.*

$$\text{grad}\left[\text{div}\ \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] - \left[\text{div grad}\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \omega^2\mu\varepsilon\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-3]}$$

Finally, we make use of Eq. [I-1c] and obtain

$$\left[\text{div grad}\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) + k^2\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \ {}^2\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) + k^2\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-4a]}$$

where $k^2 = \omega^2\mu\varepsilon$.

Similarly, starting from Eq. [I-1b] we obtain

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† That is $\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\vec{\mathbf{a}}\ \ \vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\right)$

$$\left[\text{div grad}\right]\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)+k^2\,\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)= \ {}^2\,\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)+k^2\,\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[I-4b]}$$

These are the Helmholtz equations which are the appropriate starting point for the analysis
of many problems in electromagnetic propagation. We have already used them to study pla
ne wave propagation in uniform dielectrics.

Treatment of Guided Waves:

While the Helmholtz approach is a valuable starting point in many instances, we take a slig
htly different tack in analyzing guided wave problems. The important point of departure is
that we assume the waves are propagating in the **z**-direction along a uniform guided wave s
tructure and, thus, we may write in all differential operators in terms of transverse and long
itudinal components by taking

$$\frac{\partial}{\partial z}\{\text{any field variable}\}=\gamma\{\text{any field variable}\} \qquad \text{[I-5a]}$$

so that

$$\vec{\ }\{\text{any field variable}\}=\vec{\ }_{\text{tr}}\{\text{any field variable}\}+\hat{\mathbf{z}}\gamma\{\text{any field variable}\} \qquad \text{[I-5b]}$$

where, for example and in particular,

$$\vec{\ }_{\text{tr}}=\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}\ . \qquad \text{[I-6]}$$

Furthermore, it is also useful to resolve the fields into transverse and longitudinal compone
nts -- *viz*.

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\vec{\mathbf{E}}_{\text{tr}}(\vec{\mathbf{r}},\omega)+\hat{\mathbf{z}}\,\mathbf{E}_z(\vec{\mathbf{r}},\omega) \qquad \text{[I-7a]}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=\vec{\mathbf{H}}_{\text{tr}}(\vec{\mathbf{r}},\omega)+\hat{\mathbf{z}}\,\mathbf{H}_z(\vec{\mathbf{r}},\omega) \qquad \text{[I-7b]}$$

Thus, we may rewrite Maxwell's equations and resolve them into transverse and longitudin
al components -- *viz*. Eq. [I-1a] becomes

$$\left\{\vec{\ }_{\text{tr}}+\hat{\mathbf{z}}\gamma\right\}\times\left\{\vec{\mathbf{E}}_{\text{tr}}(\vec{\mathbf{r}},\omega)+\hat{\mathbf{z}}\,\mathbf{E}_z(\vec{\mathbf{r}},\omega)\right\}=-j\,\omega\,\mu\left\{\vec{\mathbf{H}}_{\text{tr}}(\vec{\mathbf{r}},\omega)+\hat{\mathbf{z}}\,\mathbf{H}_z(\vec{\mathbf{r}},\omega)\right\} \qquad \text{[I-8a]}$$

which has transverse components

$$\gamma\ \hat{\mathbf{z}} \times \vec{\mathbf{E}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) - \hat{\mathbf{z}} \times \vec{\ }_{\mathrm{tr}}\ \mathbf{E}_z(\vec{\mathbf{r}},\omega) = -j\omega\mu\,\vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-8b]}$$

and longitudinal component

$$\vec{\ }_{\mathrm{tr}} \times \vec{\mathbf{E}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) = -j\omega\mu\,\mathbf{H}_z(\vec{\mathbf{r}},\omega) \qquad \text{[I-8c]}$$

and Eq. [I-1a] becomes

$$\left\{\vec{\ }_{\mathrm{tr}} + \hat{\mathbf{z}}\gamma\right\} \times \left\{\vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) + \hat{\mathbf{z}}\ \mathbf{H}_z(\vec{\mathbf{r}},\omega)\right\} = +j\omega\varepsilon\left\{\vec{\mathbf{E}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) + \hat{\mathbf{z}}\ \mathbf{E}_z(\vec{\mathbf{r}},\omega)\right\} \qquad \text{[I-9a]}$$

which has transverse components

$$\gamma\ \hat{\mathbf{z}} \times \vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) - \hat{\mathbf{z}} \times \vec{\ }_{\mathrm{tr}}\ \mathbf{H}_z(\vec{\mathbf{r}},\omega) = +j\omega\varepsilon\,\vec{\mathbf{E}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-9b]}$$

and longitudinal component

$$\vec{\ }_{\mathrm{tr}} \times \vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) = +j\omega\varepsilon\,\mathbf{E}_z(\vec{\mathbf{r}},\omega) \qquad \text{[I-9c]}$$

Now multiply the **first transverse equation** -- *i.e.* Eq. [I-8b] -- by $j\omega\varepsilon$

$$\gamma\ \hat{\mathbf{z}} \times \left\{j\omega\varepsilon\ \vec{\mathbf{E}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega)\right\} - j\omega\varepsilon\ \hat{\mathbf{z}} \times \vec{\ }_{\mathrm{tr}}\ \mathbf{E}_z(\vec{\mathbf{r}},\omega) = \omega^2\varepsilon\mu\ \vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega)$$

and substitute from the **second transverse equation** -- *i.e.* Eq. [I-9b]

$$\gamma\ \hat{\mathbf{z}} \times \left\{\gamma\ \hat{\mathbf{z}} \times \vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) - \hat{\mathbf{z}} \times \vec{\ }_{\mathrm{tr}}\ \mathbf{H}_z(\vec{\mathbf{r}},\omega)\right\} - j\omega\varepsilon\ \hat{\mathbf{z}} \times \vec{\ }_{\mathrm{tr}}\ \mathbf{E}_z(\vec{\mathbf{r}},\omega) = \omega^2\varepsilon\mu\,\vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega)$$

Finally, using the "abc = bac - cab" rule† we find

$$\left\{k^2 + \gamma^2\right\}\ \vec{\mathbf{H}}_{\mathrm{tr}}(\vec{\mathbf{r}},\omega) = \gamma\ \vec{\ }_{\mathrm{tr}}\ \mathbf{H}_z(\vec{\mathbf{r}},\omega) - j\omega\varepsilon\ \hat{\mathbf{z}} \times \vec{\ }_{\mathrm{tr}}\ \mathbf{E}_z(\vec{\mathbf{r}},\omega) \qquad \text{[I-10a]}$$

Similarly, multiplying the **second transverse equation** -- *i.e.* Eq. [I-9b] -- by $-j\omega\mu$

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† That is $$\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}(\vec{\mathbf{a}}\ \ \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}}\ \ \vec{\mathbf{b}})$$

$$\gamma\,\hat{\mathbf{z}}\times\left\{ j\omega\varepsilon\,\vec{\mathbf{E}}_{tr}(\vec{\mathbf{r}},\omega)\right\} - j\omega\varepsilon\,\hat{\mathbf{z}}\times\vec{\nabla}_{tr}\,\mathbf{E}_z(\vec{\mathbf{r}},\omega) = \omega^2\varepsilon\mu\,\vec{\mathbf{H}}_{tr}(\vec{\mathbf{r}},\omega)$$

and substituting from the **first transverse equation** -- *i.e.* Eq. [I-8b] -- we obtain

$$\left\{k^2+\gamma^2\right\}\vec{\mathbf{E}}_{tr}(\vec{\mathbf{r}},\omega) = \gamma\,\vec{\nabla}_{tr}\,\mathbf{E}_z(\vec{\mathbf{r}},\omega) + j\omega\mu\,\hat{\mathbf{z}}\times\vec{\nabla}_{tr}\,\mathbf{H}_z(\vec{\mathbf{r}},\omega) \qquad \text{[I-10b]}$$

In summary, these equations say: I**f we know the longitudinal electric and magnetic fields, we know everything these is to know!** But how are we to know the longitudinal fields? Answer: The longitudinal fields must, of course, obey the Helmholtz equations, which may now be written in the form

$$\nabla^2_{tr}\,\mathbf{E}_z(\vec{\mathbf{r}},\omega) + \left\{k^2+\gamma^2\right\}\mathbf{E}_z(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-11a]}$$

$$\nabla^2_{tr}\,\mathbf{H}_z(\vec{\mathbf{r}},\omega) + \left\{k^2+\gamma^2\right\}\mathbf{H}_z(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-11b]}$$

where, for example and in particular,

$$\begin{aligned}\nabla^2_{tr} &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\end{aligned} \quad . \qquad \text{[I-12]}$$

II. TEM Waveguide Modes:

The argument above is incomplete in the sense that one important case has been implicitly overlooked - *i.e.* **the case when both longitudinal fields are zero**. When does this happen? Answer: when

$$k^2+\gamma^2 = 0 \qquad \text{or} \qquad \gamma = \pm j\,k \qquad \text{[II-1]}$$

Therefore, **TEM waves always propagate at the speed of light in the dielectric** and satisfy the equations

$$\nabla^2_{tr}\,\vec{\mathbf{E}}_{tr}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[II-2a]}$$

$$\nabla_{tr}^{2}\, \vec{\mathbf{H}}_{tr}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[II-2b]}$$

An important particular example of such **TEM waveguide modes** are those derived fro
m the logarithmic potential - *viz*.

$$\varphi(\vec{\mathbf{r}},\omega) = \frac{A}{2}\ln\left[(x+a)^2 + (y+b)^2\right] \qquad \text{[II-3]}$$

which includes the TEM modes on coaxial and two-wire lines.

III. Conducting Wall Tube Waveguides:

For the nearly realizable case of ***perfectly conducting walls*** we have the simplest possi
ble situation the longitudinal fields satisfy the Helmholtz equations -- *viz*. Eqs [I-11] -- in t
he internal dielectric region and the following the boundary conditions on the walls:

$$\mathbf{E}_z(\vec{\mathbf{r}}_{wall},\omega) = 0 \qquad \text{[III-1a]}$$

$$\hat{\mathbf{z}} \times \vec{\nabla}_{tr}\, \mathbf{H}_z(\vec{\mathbf{r}}_{wall},\omega) = 0 \qquad \text{[III-1b]}$$

A. Parallel Plate Waveguides

TM-Modes (E-Waves) are derived from

$$\mathbf{E}_z^{m}(\vec{\mathbf{r}},\omega) = A_m \sin\left[\frac{m\pi x}{d}\right]\exp[\gamma_m z] \qquad \text{[III-2a]}$$

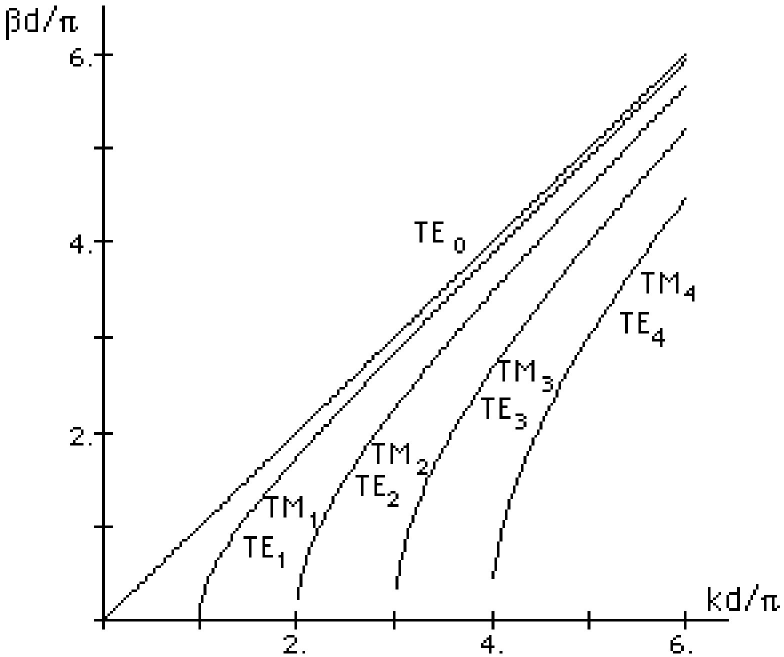
where $k^2 + \gamma_m^{\,2} = \left(\frac{m\pi}{d}\right)^2$ and thus $\gamma_m = j\beta_m = \sqrt{\left(\frac{m\pi}{d}\right)^2 - k^2}$.

TE-Modes (H-Waves) are derived from

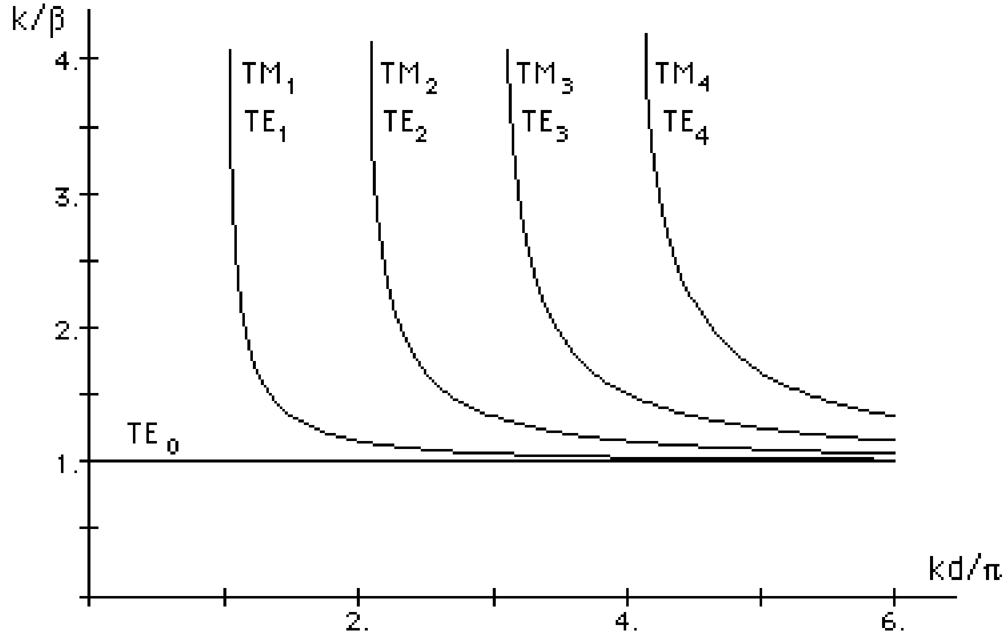
$$\mathbf{H}_z^{n}(\vec{\mathbf{r}},\omega) = B_n \cos\left[\frac{n\pi x}{d}\right]\exp[\gamma_n z] \qquad \text{[III-2b]}$$

where $k^2 + \gamma_n^{\,2} = \left(\frac{m\pi}{d}\right)^2$ and thus $\gamma_n = j\beta_n = \sqrt{\left(\frac{m\pi}{d}\right)^2 - k^2}$.

Dispersion Curves for a Parallel Plate Waveguide



"β−ω **Diagram"**



Dispersion of phase velocity

From Eqs. [I-10], we may write the complete transverse fields <u>for a given order</u>

$$\vec{\mathbf{E}}_{tr}^{m}(\vec{\mathbf{r}},\omega) = \left[k^2 + \gamma_m{}^2\right]^{-1}\left\{\gamma_m \nabla_{tr}\left[A_m \sin\left(\frac{m\pi x}{d}\right)\right] + j\omega\mu\,\hat{\mathbf{z}}\times\nabla_{tr}\left[B_m \cos\left(\frac{m\pi x}{d}\right)\right]\right\}\exp\left[\gamma_m z\right] \quad \text{[III-3a]}$$

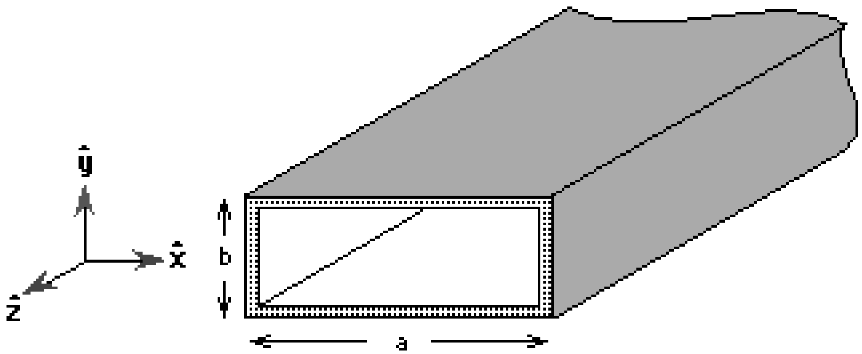
$$\vec{\mathbf{H}}_{tr}^{m}(\vec{\mathbf{r}},\omega) = \left[k^2 + \gamma_m{}^2\right]^{-1}\left\{\gamma_m \nabla_{tr}\left[B_m \cos\left(\frac{m\pi x}{d}\right)\right] - j\omega\varepsilon\,\hat{\mathbf{z}}\times\nabla_{tr}\left[A_m \sin\left(\frac{m\pi x}{d}\right)\right]\right\}\exp\left[\gamma_m z\right] \quad \text{[III-3a]}$$

which *simplify* to

$$\vec{\mathbf{E}}_{tr}^{m}(\vec{\mathbf{r}},\omega) = \left[\frac{d}{m\pi}\right]\left\{j\beta_m A_m \cos\left(\frac{m\pi x}{d}\right)\hat{\mathbf{x}} - j\omega\mu B_m \sin\left(\frac{m\pi x}{d}\right)\hat{\mathbf{y}}\right\}\exp\left[j\beta_m z\right] \quad \text{[III-4a]}$$

$$\vec{\mathbf{H}}^{m}_{tr}(\vec{\mathbf{r}},\omega) = \left[\frac{d}{m\pi}\right]\left\{-j\,\beta_m\,B_m\,\sin\left(\frac{m\pi x}{d}\right)\hat{\mathbf{x}} - j\,\omega\,\varepsilon\,A_m\,\cos\left(\frac{m\pi x}{d}\right)\hat{\mathbf{y}}\right\}\exp\left[j\,\beta_m\,z\right] \qquad \text{[III-4b]}$$

B. Rectangular Waveguides



TM-Modes (E-Waves) are derived from

$$\mathbf{E}^{mn}_{z}(\vec{\mathbf{r}},\omega) = A_{mn}\,\sin\left[\frac{m\pi x}{a}\right]\sin\left[\frac{n\pi y}{b}\right]\exp\left[\gamma_{mn}\,z\right] \qquad \text{[III-5a]}$$

where $k^2 + \gamma_m{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

and thus $\gamma_{mn} = j\,\beta_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$.

TE-Modes (H-Waves) are derived from

$$\mathbf{H}^{mn}_{z}(\vec{\mathbf{r}},\omega) = B_{mn}\,\cos\left[\frac{m\pi x}{a}\right]\cos\left[\frac{n\pi y}{b}\right]\exp\left[\gamma_{mn}\,z\right] \qquad \text{[III-5b]}$$

where, again, $k^2 + \gamma_m{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

and $$\gamma_{mn} = j\,\beta_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$$ [III-6]

Dispersion Curves for a Rectangular Waveguide

(Most common or standard configuration where $a = 2b$)